

A FAMILY OF CONFORMALLY FLAT HAMILTONIAN-MINIMAL LAGRANGIAN TORI IN \mathbb{CP}^3

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ABSTRACT. In this paper by reduction we construct a family of conformally flat Hamiltonian-minimal Lagrangian tori in \mathbb{CP}^3 as the image of the composition of the Hopf map $\mathcal{H} : \mathbb{S}^7 \rightarrow \mathbb{CP}^3$ and a map $\psi : \mathbb{R}^3 \rightarrow \mathbb{S}^7$ with certain conditions.

MSC(2000): 05C42, 53D12, 35Q51;

Key words: conformally flat, Hamiltonian-minimal Lagrangian tori

1. INTRODUCTION

A Lagrangian submanifold of a Kähler manifold is said to be Hamiltonian-minimal (briefly, H-minimal) if it is a critical point of the volume under Hamiltonian deformations. This notion was introduced by Oh in [17], who also gave an example, that is, Clifford tori in \mathbb{C}^n with standard Hermitian metric

$$\mathbb{S}^1(r_1) \times \cdots \times \mathbb{S}^1(r_n) \subset \mathbb{C}^n,$$

where $\mathbb{S}^1(r_j)$ is a circle of radius r_j in \mathbb{C} .

In [7], Hélein and Romon studied a general construction of H-minimal tori in \mathbb{C}^2 from the point of view of completely integrable systems. They provided new explicit nontrivial examples of H-minimal Lagrangian tori which include the examples previously constructed by Castro and Urbano in \mathbb{C}^2 [3]. A similar construction has been generalized to the cases of Hermitian symmetric spaces, e.g., in \mathbb{CP}^2 , see [8, 11, 10] for details. However, in the non-flat cases, the underlying equations are no longer linear, which makes the problem much harder. Although in [10], Ma introduced a spectral parameter $\lambda \in \mathbb{S}^1$, as she pointed out, a description of H-minimal Lagrangian tori in \mathbb{CP}^2 in terms of theta functions seems to be possible. But owing to this spectral parameter $\lambda \notin \mathbb{C}$, thus it is still open about the integrability of this problem in classical sense. In [13, 14] it is shown that if a Lagrangian conformal map from \mathbb{R}^2 to \mathbb{CP}^2 is given as composition of maps $\varphi := \mathcal{H} \circ \psi : \mathbb{R}^2 \rightarrow S^5 \rightarrow \mathbb{CP}^2$, where \mathcal{H} is the Hopf map, then components of φ satisfy Shrödinger equation

$$\Delta \varphi_j + i(\theta_x \partial_x \varphi_j + \theta_y \partial_y \varphi_j) + 4e^u \varphi_j = 0,$$

where $ds^2 = 2e^u(dx^2 + dy^2)$ is an induced metric and $\theta(x, y)$ is Lagrangian angle. In the case of H-minimal Lagrangian tori θ is a linear function. So in order to construct finite gap tori it is necessary to use spectral data of finite gap Shrödiger operators, see [6], [18] for details.

In [2], Castro and Urbano constructed a family of minimal Lagrangian tori in \mathbb{CP}^2 , which are characterized by their invariance under a one-parameter group of holomorphic isometries of \mathbb{CP}^2 . By using this idea, in [13, 9], they independently reduced this problem to a one dimensional system and obtained an equivariant solution in terms of elliptic functions, and then constructed H-minimal Lagrangian tori in \mathbb{CP}^2 .

In high dimensional case one of the present authors constructed some examples of H-minimal and minimal Lagrangian immersions and embeddings in \mathbb{C}^n and \mathbb{CP}^n . Recently we can find some of works about this topic, see [1, 4, 5] and references therein. But it is far from the complete characterization.

In this paper we address to construct a family of conformally flat H-minimal Lagrangian tori in \mathbb{CP}^3 by reduction methods, which generalizes the results in [13, 15, 9], with the metric

$$ds^2 = e^u(dx^2 + dy^2 + dz^2).$$

In \mathbb{CP}^3 , it seems to be a little harder. We thus restrict to discuss a very special case, that is, $u = u(z)$ and the Lagrangian angle $\theta = ax + by$, where a and b are arbitrary real constants.

2. PRELIMINARIES

In this section, we review some well-known facts without proofs and sketch our strategy about the construction of H-minimal Lagrangian cone or tori in \mathbb{C}^4 and \mathbb{CP}^3 .

2.1. Notations. Let \mathbb{C}^4 be the canonical complex space of dimension 4 endowed with an Hermitian product $\langle u, v \rangle = \sum_{k=1}^4 u_k \bar{v}_k$. Let us denote $\omega = \text{Im} \langle \cdot, \cdot \rangle$ and $(\cdot, \cdot) = \text{Re} \langle \cdot, \cdot \rangle$. Let $\psi : \mathbb{R}^3 \rightarrow \mathbb{S}^7$ be an oriented Lagrangian immersion L , i.e. $\psi^* \omega = 0$, where \mathbb{S}^7 is the unit sphere in \mathbb{C}^4 . Wolfson in [19] introduced a Lagrangian angle θ and obtained a criterion of H-minimality of L in terms of θ , that is, *the Lagrangian immersion L is H-minimal if and only if the Lagrangian angle θ is a harmonic function on L* . The Lagrangian angle θ of L in \mathbb{C}^4 is defined by the formula

$$e^{i\theta(p)} = dz_1 \wedge \cdots \wedge dz_4(\Psi), \quad p \in L,$$

where $z_j, j = 1, \dots, 4$ are coordinates on \mathbb{C}^4 and Ψ is an orthonormal tangent frame at $p \in L$ with the same orientation of L . For the general case, see [19] for details.

Let $\mathcal{H} : S^7 \rightarrow \mathbb{CP}^3$ be the Hopf map. An induced Hermitian product on \mathbb{CP}^3 is called to be the Fubini-Study metric defined by $\langle \zeta_1, \zeta_2 \rangle := \langle \tilde{\zeta}_1, \tilde{\zeta}_2 \rangle$, where $\zeta_i, i = 1, 2$ are tangent to \mathbb{CP}^3 and $\tilde{\zeta}_i$ are the corresponding horizontal lifting by \mathcal{H} . Let \mathcal{C} be a Lagrangian cone in \mathbb{C}^4 with the vertex at the origin. It follows from the definition of $\langle \cdot, \cdot \rangle$ that $\mathcal{H}(\tilde{\mathcal{C}})$ is a Lagrangian submanifold in \mathbb{CP}^3 , where $\tilde{\mathcal{C}} = \mathcal{C} \cap S^7$. Moreover, if the cone \mathcal{C} is H-minimal in \mathbb{C}^4 , then $\mathcal{H}(\tilde{\mathcal{C}})$ is also H-minimal in \mathbb{CP}^3 , see [12, 1] for details.

2.2. On conformally flat Lagrangian immersions. In the following we only consider conformally flat immersions in \mathbb{C}^4 and \mathbb{CP}^3 .

Let $\psi = (\psi^1, \psi^2, \psi^3, \psi^4) : \mathbb{R}^3 \rightarrow \mathbb{S}^7 \subset \mathbb{C}^4$ be an oriented immersion with a conformally flat metric

$$ds^2 = e^{u(x,y,z)}(dx^2 + dy^2 + dz^2)$$

satisfying the following properties

$$\langle \psi, \psi_x \rangle = \langle \psi, \psi_y \rangle = \langle \psi, \psi_z \rangle = 0,$$

$$\langle \psi_x, \psi_y \rangle = \langle \psi_y, \psi_z \rangle = \langle \psi_z, \psi_x \rangle = 0,$$

by the above arguments, thus $\mathcal{H} \circ \psi$ is a Lagrangian immersion in \mathbb{CP}^3 and

$$(2.1) \quad \Phi = (\psi, e^{-u}\psi_x, e^{-u}\psi_y, e^{-u}\psi_z)^t \in \text{U}(4)$$

and the Lagrangian angle $\theta(x, y, z)$ is given by

$$(2.2) \quad e^{i\theta} = \det(\Phi).$$

By using (2.1) and (2.2), we have

$$(2.3) \quad \Psi = (e^{i\theta}\psi, e^{-u}\psi_x, e^{-u}\psi_y, e^{-u}\psi_z)^t \in \text{SU}(4)$$

and denote

$$(2.4) \quad \mathcal{U} := \Psi_x \Psi^{-1}, \quad \mathcal{V} := \Psi_y \Psi^{-1}, \quad \mathcal{W} := \Psi_z \Psi^{-1} \in \text{SU}(4).$$

The compatibility condition of (2.4) is

$$(2.5) \quad \mathcal{U}_y - \mathcal{V}_x + [\mathcal{U}, \mathcal{V}] = 0, \quad \mathcal{V}_z - \mathcal{W}_y + [\mathcal{V}, \mathcal{W}] = 0, \quad \mathcal{W}_x - \mathcal{U}_z + [\mathcal{W}, \mathcal{U}] = 0.$$

Moreover, if $\Delta\theta(x, y, z) = 0$, then the immersion $\mathcal{H} \circ \psi$ is a conformally flat H-minimal Lagrangian immersion in \mathbb{CP}^3 , where Δ is the corresponding Laplacian operator given by the formula

$$\Delta := -e^{-u(z)} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{u_x}{2} \frac{\partial}{\partial x} + \frac{u_y}{2} \frac{\partial}{\partial y} + \frac{u_z}{2} \frac{\partial}{\partial z} \right].$$

Conversely, given $\text{SU}(4)$ -valued \mathcal{U} , \mathcal{V} , \mathcal{W} satisfying (2.5), we solve the system (2.4) with (2.3) for $\Psi \in \text{SU}(4)$ and then obtain the immersion $\psi : \mathbb{R}^3 \rightarrow \mathbb{S}^7$ and the Lagrangian angle θ . If $\Delta\theta = 0$ and ψ satisfies certain periodic conditions, then the immersion $\mathcal{H} \circ \psi$ gives a conformally flat H-minimal Lagrangian torus in \mathbb{CP}^3 . Notice that for general case, this method does not work.

3. CONFORMALLY FLAT H-MINIMAL LAGRANGIAN TORI IN \mathbb{CP}^3

In this section, by using the above method, we will construct a special class of conformally flat H-minimal Lagrangian tori in \mathbb{CP}^3 with $u = u(z)$ and the Lagrangian angle $\theta = ax + by$, where a and b are arbitrary real constants.

3.1. The first step is to choose \mathcal{U} , \mathcal{V} and \mathcal{W} . By a direct calculation, we have the following lemma.

Lemma 3.1. *Suppose $\mathcal{U} = (u_{kl})$, $\mathcal{V} = (v_{kl})$, $\mathcal{W} = (w_{kl})$ are $SU(4)$ -valued matrices satisfying (2.4) and u_{kl}, v_{kl}, w_{kl} depend only one variable z for $2 \leq k, l \leq 4$, then they must be the following form*

$$\mathcal{U} = \begin{pmatrix} ia & e^{i\theta+u} & 0 & 0 \\ -e^{-i\theta+u} & -i(a+c_1) & ic_2 & ic_3e^{-3u} - u' \\ 0 & ic_2 & ic_1 & 0 \\ 0 & ic_3e^{-3u} + u' & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{V} = \begin{pmatrix} ib & 0 & e^{i\theta+u} & 0 \\ 0 & ic_2 & ic_1 & 0 \\ -e^{-i\theta+u} & ic_1 & -i(b+c_2) & ic_3e^{-3u} - u' \\ 0 & 0 & ic_3e^{-3u} + u' & 0 \end{pmatrix},$$

and

$$\mathcal{W} = \begin{pmatrix} 0 & 0 & 0 & e^{i\theta+u} \\ 0 & ic_3e^{-3u} & 0 & 0 \\ 0 & 0 & ic_3e^{-3u} & 0 \\ -e^{-i\theta+u} & 0 & 0 & -2ic_3e^{-3u} \end{pmatrix},$$

where $u = u(z)$ satisfies the equation

$$(3.1) \quad u'^2 + e^{2u} + c_3^2 e^{-6u} - \mathfrak{C} = 0, \quad u' = \frac{du}{dz}.$$

and c_1, c_2, c_3 are arbitrary real constants and $\mathfrak{C} = ac_1 + bc_2 + 2c_1^2 + 2c_2^2$.

3.2. The second step is to solve the following system

$$(3.2) \quad \Psi_x = \mathcal{U}\Psi, \quad \Psi_y = \mathcal{V}\Psi, \quad \Psi_z = \mathcal{W}\Psi.$$

Write

$$(3.3) \quad \Psi = (e^{i\theta}\psi, e^{-u}\psi_x, e^{-u}\psi_y, e^{-u}\psi_z)^t,$$

where $\psi = \psi(x, y, z)$ is a smooth function. By using (3.3), the system (3.2) can be rewritten as

$$(3.4) \quad \psi_{xz} - (u' + ic_3e^{-3u})\psi_x = 0,$$

$$(3.5) \quad \psi_{yz} - (u' + ic_3e^{-3u})\psi_y = 0,$$

$$(3.6) \quad \psi_{xy} - i(c_2\psi_x + ic_1\psi_y) = 0,$$

and

$$(3.7) \quad \psi_{xx} + e^{2u}\psi + i(a+c_1)\psi_x - ic_2\psi_y + ((u' - ic_3e^{-3u})\psi_z = 0,$$

$$(3.8) \quad \psi_{yy} + e^{2u}\psi - ic_1\psi_x + i(c_2+b)\psi_y + ((u' - ic_3e^{-3u})\psi_z = 0,$$

$$(3.9) \quad \psi_{zz} + e^{2u}\psi + (2ic_3e^{-3u} - u')\psi_z = 0.$$

From (3.4) and (3.5), we know that ψ must be of the form

$$(3.10) \quad \psi = P(z)\varphi(x, y) + Q(z), \quad \varphi(x, y) \neq \text{constant},$$

and then (3.4) and (3.5) reduce to

$$(3.11) \quad P(z) - (u' + ic_3e^{-3u})P(z) = 0.$$

The solution of (3.11) is

$$(3.12) \quad P(z) = a_1 e^{u+ic_3 \int e^{-3u} dz}, \quad a_1 \in \mathbb{R}.$$

Substituting (3.10) and (3.12) into (3.6), we get

$$(3.13) \quad \varphi_{xy} - i(c_2 \varphi_x + c_1 \varphi_y) = 0.$$

Substituting (3.10) into (3.7) and (3.8), and then using (3.1) and (3.13), we get the following

$$(3.14) \quad Q'(z) = \frac{e^{5u} Q(z)}{ic_3 - u' e^{3u}},$$

$$(3.15) \quad \varphi_{xx} + i(a + c_1) \varphi_x - ic_2 \varphi_y + \mathfrak{C} \varphi = 0,$$

$$(3.16) \quad \varphi_{yy} - ic_1 \varphi_x + i(b + c_2) \varphi_y + \mathfrak{C} \varphi = 0.$$

Write

$$(3.17) \quad Q(z) = H(z) e^{iG(z)},$$

where $G(z)$ and $H(z)$ are real smooth functions. By differentiating (3.1), we obtain

$$(3.18) \quad u'' + e^{2u} - 3c_3^2 e^{-6u} = 0.$$

By using (3.18), and separating the real part and the imaginary part of (3.14), we obtain

$$H'(z)(\mathfrak{C} - e^{2u}) + u' e^{2u} H(z) = 0, \quad G'(z)(e^{2u} - \mathfrak{C}) - c_3 e^{-u} = 0,$$

thus

$$H(z) = a_2 \sqrt{\mathfrak{C} - e^{2u}}, \quad a_2 \in \mathbb{R}, \quad G(z) = \int \frac{c_3 e^{-u}}{e^{2u} - \mathfrak{C}} dz + a_3, \quad a_3 \in \mathbb{R}.$$

From (3.13), without loss of generality, we could assume that $\varphi(x, y)$ has the form

$$\varphi(x, y) = \sum a_{\alpha_j \beta_j} e^{i(x\alpha_j + y\beta_j)}, \quad \beta_j = \frac{c_2 \alpha_j}{\alpha_j - c_1}, \quad a_{\alpha_j \beta_j} \in \mathbb{C}, \quad a_{00} = 0.$$

It follows from (3.17), (3.15) and (3.16) that $\alpha = \alpha_j$ is a root of the equation

$$\alpha^3 + a\alpha^2 - \mathfrak{B}\alpha + c_1 \mathfrak{C} = 0.$$

where $\mathfrak{B} = 2c_1 a + 3c_1^2 + c_2 b + 3c_2^2$.

Notice that up to now we only use (3.4)—(3.8) to obtain an explicit form of ψ as follows

$$(3.19) \quad \begin{aligned} \psi(x, y, z) &= \sum a_1 a_{\alpha_j \beta_j} e^{u(z) + ic_3 \int e^{-3u(z)} dz} e^{i(x\alpha_j + y\beta_j)} \\ &+ a_2 e^{ia_3} \sqrt{\mathfrak{C} - e^{2u(z)}} e^{ic_3 \int \frac{e^{-u(z)}}{e^{2u(z)} - \mathfrak{C}} dz}. \end{aligned}$$

Furthermore, it is easy to check that this function ψ also satisfies (3.9). Thus we solve the system (3.2). Summarizing the above discussions, we have the following proposition.

Proposition 3.2. *If we suppose that $u = u(z)$ is a smooth solution of $u(z)'^2 + e^{2u(z)} + c_3^2 e^{-6u(z)} - \mathfrak{C} = 0$; and α is a root of the equation*

$$(3.20) \quad \alpha^3 + a\alpha^2 - \mathfrak{B}\alpha + c_1\mathfrak{C} = 0$$

Then $\Psi = (e^{i\theta}\psi, e^{-u}\psi_x, e^{-u}\psi_y, e^{-u}\psi_z)^t$ is a solution of the system (3.2) with $\theta = ax + by$ and

$$\psi(x, y, z) = \kappa_1 e^{i(\alpha x + \beta y)} P(z) + \kappa_2 Q(z), \quad \beta = \frac{c_2 \alpha}{\alpha - c_1},$$

where κ_1 and κ_2 are arbitrary complex constants and

$$P(z) = e^{u(z) + ic_3 \int e^{-3u(z)} dz}, \quad Q(z) = \sqrt{\mathfrak{C} - e^{2u(z)}} e^{ic_3 \int \frac{e^{-u(z)}}{e^{2u(z)} - \mathfrak{C}} dz}.$$

3.3. Main results. We now state our main theorem.

Theorem 3.3. *Suppose that the equation (3.20) has three distinct roots, denoted by α_1, α_2 and α_3 . Write $\beta_j = \frac{c_2 \alpha_j}{\alpha_j - c_1}$, $j = 1, 2, 3$. Then the map $\mathcal{H} \circ \psi : \mathcal{R}^3 \rightarrow \mathbb{CP}^3$ defines a conformally flat H -minimal Lagrangian immersion in \mathbb{CP}^3 , where $\mathcal{H} : \mathbb{S}^7 \rightarrow \mathbb{CP}^3$ is the Hopf map and the map $\psi : \mathcal{R}^3 \rightarrow \mathbb{S}^7 \subset \mathbb{C}^4$ is given by the formula*

$$\psi = (\gamma_1 P(z) e^{i(x\alpha_1 + y\beta_1)}, \gamma_2 P(z) e^{i(x\alpha_2 + y\beta_2)}, \gamma_3 P(z) e^{i(x\alpha_3 + y\beta_3)}, \gamma_4 Q(z)).$$

Here $\gamma_4 = \sqrt{\frac{1}{\mathfrak{C}}}$ and

$$\gamma_1 = \sqrt{\frac{\mathfrak{C} + \alpha_2 \alpha_3}{\mathfrak{C}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}}, \quad \gamma_2 = \sqrt{\frac{\mathfrak{C} + \alpha_1 \alpha_3}{\mathfrak{C}(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)}}, \quad \gamma_3 = \sqrt{\frac{\mathfrak{C} + \alpha_1 \alpha_2}{\mathfrak{C}(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}}.$$

Proof. It suffices to check that

$$\Psi = (e^{i\theta}\psi, e^{-u}\psi_x, e^{-u}\psi_y, e^{-u}\psi_z)^t \in \text{SU}(4).$$

By using (3.20), we have

$$(3.21) \quad \alpha_1 + \alpha_2 + \alpha_3 = -a, \quad \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 = -\mathfrak{B}, \quad \alpha_1 \alpha_2 \alpha_3 = -\mathfrak{C}.$$

It follows from (3.21) and the explicit forms of γ_j that

$$\begin{aligned} \sum_{j=1}^3 \gamma_j^2 &= \gamma_4^2, & \sum_{j=1}^3 \gamma_j^2 \alpha_j &= 0, & \sum_{j=1}^3 \gamma_j^2 \alpha_j^2 &= 1, \\ \sum_{j=1}^3 \gamma_j^2 \alpha_j \beta_j &= 0, & \sum_{j=1}^3 \gamma_j^2 \beta_j &= 0, & \sum_{j=1}^3 \gamma_j^2 \beta_j^2 &= 1. \end{aligned}$$

These identities yield that

$$\begin{aligned}
\langle \psi, \psi \rangle &= 1, \langle \psi_x, \psi_x \rangle = \langle \psi_y, \psi_y \rangle = e^{2u}, \\
\langle \psi, \psi_x \rangle &= \langle \psi, \psi_y \rangle = \langle \psi, \psi_z \rangle = 0, \\
\langle \psi_x, \psi_y \rangle &= \langle \psi_y, \psi_z \rangle = \langle \psi_z, \psi_x \rangle = 0, \\
\langle \psi_z, \psi_z \rangle &= P'(z) \overline{P'(z)} \sum_{j=1}^3 \gamma_j^2 + \gamma_4^2 Q'(z) \overline{Q'(z)} \\
&= \gamma_4^2 [e^{2u}(u'2 + c_3^2 e^{-6u}) + \frac{e^{4u}(u'2 + c_3^2 e^{-6u})}{\mathfrak{C} - e^{2u}}] \\
&= e^{2u}. \quad \text{by using (3.1)}
\end{aligned}$$

That is to say, $\Psi \in \text{SU}(4)$. Thus we complete the proof of the theorem. \square

We finish this section to discuss how to obtain conformally flat H-minimal Lagrangian tori in \mathbb{CP}^3 .

Notice that in (3.1) if we make the following change

$$(3.22) \quad u = u(z) := -\log(2\sqrt{-q(z)}),$$

then we have

$$(3.23) \quad q'(z)^2 = 256c_3^2 q(z)^5 + 4\mathfrak{C}q(z)^2 + q(z).$$

Thus if we choose three real constants c_1 , c_2 and c_3 such that the equation

$$256c_3^2 t^5 + 4\mathfrak{C}t^2 + t = 0$$

has two negative roots and does not have multiple roots, then this assures that (3.23) has a smooth periodic solution of the period τ , see [16] for details. It follows from (3.22) that so is (3.1). We here remark that in this case $\mathfrak{C} = ac_1 + bc_2 + 2c_1^2 + 2c_2^2 > 0$.

We next discuss the condition such that the function ψ in (3.19) is a periodic function of x, y, z respectively. According to the form of ψ in (3.19), if we assume that $c_1 \in \mathbb{Q}$ and α_2, α_2 and α_3 are three distinct rational roots of (3.20), then ψ is periodic w.r.t. x and y . Notice that $u(z + \tau) = u(z)$ and there exists a periodic function $h(z)$ of the periodic τ such that

$$\int \frac{e^{-3u(z)}\mathfrak{C}}{e^{2u(z)} - \mathfrak{C}} dz = h(z) + z \int_0^\tau \frac{e^{-3u(z)}\mathfrak{C}}{e^{2u(z)} - \mathfrak{C}} dz.$$

This implies that if we assume

$$\frac{c_3\mathfrak{C}\tau}{2\pi} \int_0^\tau \frac{e^{-3u(z)}}{e^{2u(z)} - \mathfrak{C}} dz \in \mathbb{Q},$$

then ψ is periodic in z with the period $n\tau$ for some $n \in \mathbb{N}$.

Thus, combining with Theorem 3.3, we have

Theorem 3.4. *If we suppose that*

1. $u = u(z)$ is a periodic solution of (3.1) with the period τ ;
2. $c_1 \in \mathbb{Q}$, $\frac{c_3\mathfrak{C}\tau}{2\pi} \int_0^\tau \frac{e^{-3u(z)}}{e^{2u(z)} - \mathfrak{C}} dz \in \mathbb{Q}$;
3. α_1, α_2 and α_3 are distinct rational roots of (3.20).

Then the map $\mathcal{H} \circ \psi : \mathbb{R}^3 \rightarrow \mathbb{CP}^3$ defines a conformally flat H -minimal Lagrangian torus in \mathbb{CP}^3 .

ACKNOWLEDGMENT

A.E.Mironov is grateful to generous supports by KIAS, where part of the work was done in KIAS. The work of A.E.Mironov was partially supported by Russian Federation foundation for basic research (grant no.06-01-00094a) and grant MK-9651.2006.1 of the president of Russian Federation.

Zuo would like to thank Qing Chen and Bumsig Kim and Youjin Zhang for their constant guidance and supports. Zuo also thanks Chengmin Bai for hospitality during stay at Chern Institute of mathematics in Nankai university, where part of the work was done. The work of Zuo was partially supported by a post-doc fellowship from KIAS and the Natural Science Foundation of China (grant no. 10501043).

Both of the authors thank organizers for the hospitality during the ISLAND-3 on Islay, Scotland.

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